TORSION FUNCTORS OF LOCAL COHOMOLOGY MODULES

MOHAMMAD T. DIBAEI[1] AND ALIREZA VAHIDI[2]

Abstract. Through a study of torsion functors of local cohomology modules we improve some non-finiteness results on the top non-zero local cohomology modules with respect to an ideal.

1. Introduction

Let R be a commutative Noetherian ring with non-zero identity. We use symbols $\mathfrak{a}$, M, and X as an ideal of R, a finite (i.e. finitely generated) R–module, and an arbitrary R–module which is not necessarily finite. The ith local cohomology module of X with respect to $\mathfrak{a}$ is denoted by $\mathrm{H}^i_{\mathfrak{a}}(X)$.

For all $i \geq 0$, it is well known that $\mathrm{H}^i_{\mathfrak{m}}(M)$ is Artinian for any maximal ideal $\mathfrak{m}$ of R. In particular, $\mathrm{Hom}_R(R/\mathfrak{m}, \mathrm{H}^i_{\mathfrak{m}}(M))$ is finite. Grothendieck asked, in [6], whether a similar statement is valid if $\mathfrak{m}$ is replaced by an arbitrary ideal of R. Hartshorne gave a counterexample in [8] and raised the question whether $\mathrm{Ext}^i_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))$ is finite for all i and j, and proved this is the case when R is a complete regular local ring and $\dim(R/\mathfrak{a}) = 1$. This result was later extended to more general rings by Delfino and Marley ([4, Theorem 1]).

For an R–module X, Melkersson [11, Theorem 2.1] proved that $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is finite for all i if and only if $\mathrm{Tor}^R_i(R/\mathfrak{a}, X)$ is finite for all i. Summarizing the above results, we see that for any ideal $\mathfrak{a}$ of R with $\dim(R/\mathfrak{a}) \leq 1$, $\mathrm{Tor}^R_i(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))$ is finite for all i and j. This result inspired us to study $\mathrm{Tor}^R_i(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(X))$ in general for an arbitrary R–module X. Note that there are some attempts to study $\mathrm{Tor}^R_0(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(X))$ in [2] and $\mathrm{Tor}^R_i(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))$ in [10].

In Section 2, we present some technical results (Lemma 2.1 and Theorem 2.2) which show that, in certain situation, the torsion module $\mathrm{Tor}^R_i(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(X))$ is in a Serre subcategory of the category of R–modules. Recall that $\mathcal{S}$ is a Serre subcategory of the category of R–modules if for any exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0 \tag{1.1}$$

the module X is in $\mathcal{S}$ if and only if X' and X'' are in $\mathcal{S}$. Always, $\mathcal{S}$ stands for a Serre subcategory of the category of R–modules.

2000 *Mathematics Subject Classification.* 13D45, 13D07, 13C12.

Key words and phrases. coatomic modules, local cohomology modules, minimax modules, spectral sequences.

The research of the first author was in part supported by a grant from IPM (No. 88130126).

Section 3 consists of applications. In Corollary 3.3, we show that, for certain integer i, $\mathrm{H}^i_{\mathfrak{a}}(X)$ may not be finite, coatomic, or minimax. Recall that, an R–module X is said to be *coatomic* (resp. *minimax*) if any submodule of X is contained in a maximal submodule of X (resp. if there is a finite submodule X' of X such that X/X' is Artinian). Finally, we show that, for a positive integer n, the statement "$\mathrm{H}^i_{\mathfrak{a}}(X)$ is coatomic for all $i \geq n$" is equivalent to each of the statements "$\mathrm{H}^i_{\mathfrak{a}}(X)$ is finite for all $i \geq n$" and "$\mathrm{H}^i_{\mathfrak{a}}(X) = 0$ for all $i \geq n$"; also the statement "$\mathrm{H}^i_{\mathfrak{a}}(X)$ is minimax for all $i \geq n$" is equivalent to the statement "$\mathrm{H}^i_{\mathfrak{a}}(X)$ is Artinian for all $i \geq n$" (Corollaries 3.4 and 3.5).

2. Main result

In this section, c denotes the arithmetic rank of the ideal $\mathfrak{a}$, so that there exist elements $x_1, \cdots, x_c$ of R such that $\sqrt{\mathfrak{a}} = (x_1, \cdots, x_c)$, also $C(X)^\bullet$ denotes the Čech complex of X with respect to $x_1, \cdots, x_c$. It is well known that the ith cohomology module of $C(X)^\bullet$ is isomorphic to the ith local cohomology module $\mathrm{H}^i_{\mathfrak{a}}(X)$ (see [3, Theorem 5.1.19]).

Our method is based on the following lemma. We adopt the notation as in [12].

Lemma 2.1. *Assume that X and N are R–modules such that N is $\mathfrak{a}$-torsion. Then there is a first quadrant spectral sequence*

$$E^2_{p,q} := \mathrm{Tor}\,^R_p(N, \mathrm{H}^{c-q}_{\mathfrak{a}}(X)) \underset{p}{\Longrightarrow} \mathrm{Tor}\,^R_{p+q-c}(N, X). \tag{2.1}$$

Proof. Let $F_\bullet$ be a free resolution of N and consider the first quadrant bicomplex $\mathcal{T} = \{C(F_p \otimes_R X)^{c-q}\}$. We denote the total complex of $\mathcal{T}$ by $\mathrm{Tot}\,(\mathcal{T})$. The first filtration has E^2 term the iterated homology $H'_p H''_{p,q}(\mathcal{T})$. By [3, Theorem 5.1.19], we have

$$H''_{p,q}(\mathcal{T}) = H^{c-q}(C(F_p \otimes_R X)^\bullet) = H^{c-q}_{\mathfrak{a}}(F_p \otimes_R X) = F_p \otimes_R H^{c-q}_{\mathfrak{a}}(X).$$

Hence

$${}^{I}E^2_{p,q} = H_p(F_\bullet \otimes_R \mathrm{H}^{c-q}_{\mathfrak{a}}(X)) = \mathrm{Tor}\,^R_p(N, H^{c-q}_{\mathfrak{a}}(X)).$$

On the other hand, the second filtration has E^2 term the iterated homology $H''_p H'_{q,p}(\mathcal{T})$. We have

$$H'_{q,p}(\mathcal{T}) = H_q(C(R)^{c-p} \otimes_R F_\bullet \otimes_R X) = C(R)^{c-p} \otimes_R H_q(F_\bullet \otimes_R X) = C(\mathrm{Tor}\,^R_q(N, X))^{c-p}.$$

Thus, again by [3, Theorem 5.1.19],

$${}^{II}E^2_{p,q} = H^{c-p}(C(\mathrm{Tor}\,^R_q(N, X))^\bullet) = H^{c-p}_{\mathfrak{a}}(\mathrm{Tor}\,^R_q(N, X)).$$

Since $\mathrm{Tor}\,^R_q(N, X)$ is $\mathfrak{a}$–torsion for all q,

$${}^{II}E^2_{p,q} \cong \begin{cases} \mathrm{Tor}\,^R_q(N, X) & \text{if} \quad p = c, \\ 0 & \text{if} \quad p \neq c. \end{cases}$$

Therefore this spectral sequence collapses at the cth column and so

$$H_{p+q}(\mathrm{Tot}\,(\mathcal{T})) = {}^{II}E^2_{c,p+q-c} = \mathrm{Tor}\,^R_{p+q-c}(N, X)$$

which yields the assertion. □

It is our main object to find out when a torsion functor of a local cohomology module is in a Serre subcategory $\mathcal{S}$. Note that the following subcategories are examples of Serre subcategories of the category of R–modules: finite R–modules; Artinian R–modules; coatomic R–modules ([15]); minimax R–modules ([14]); and trivially the zero R–module. In the following theorem, we find some sufficient conditions for this purpose.

Theorem 2.2. *Suppose that X and N are R–modules such that N is $\mathfrak{a}$–torsion. Assume also that s, t are non-negative integers such that*

(i) $\operatorname{Tor}_{s-t}^{R}(N, X)$ *is in* $\mathcal{S}$*,*
(ii) $\operatorname{Tor}_{s-t+i-1}^{R}(N, \mathrm{H}_{\mathfrak{a}}^{i}(X))$ *is in* $\mathcal{S}$ *for all* i*,* $0 \leq i \leq t-1$*, and*
(iii) $\operatorname{Tor}_{s-t+i+1}^{R}(N, \mathrm{H}_{\mathfrak{a}}^{i}(X))$ *is in* $\mathcal{S}$ *for all* i*,* $t+1 \leq i \leq c$*.*

Then $\operatorname{Tor}_{s}^{R}(N, \mathrm{H}_{\mathfrak{a}}^{t}(X))$ *is in* $\mathcal{S}$*.*

Proof. We may assume that $t \leq c$. Set $u = c-t$, $n = s+u$, and consider the spectral sequence (2.1). For all $r \geq 2$, let $Z_{s,u}^{r} = \ker(E_{s,u}^{r} \longrightarrow E_{s-r,u+r-1}^{r})$ and $B_{s,u}^{r} = \operatorname{Im}(E_{s+r,u-r+1}^{r} \longrightarrow E_{s,u}^{r})$. So that we have the exact sequences:

$$0 \longrightarrow Z_{s,u}^{r} \longrightarrow E_{s,u}^{r} \longrightarrow E_{s,u}^{r}/Z_{s,u}^{r} \longrightarrow 0$$

and

$$0 \longrightarrow B_{s,u}^{r} \longrightarrow Z_{s,u}^{r} \longrightarrow E_{s,u}^{r+1} \longrightarrow 0.$$

Note that $E_{s-r,u+r-1}^{2}$ and $E_{s+r,u-r+1}^{2}$ are in $\mathcal{S}$ by assumptions (ii) and (iii), so that their subquotients $E_{s-r,u+r-1}^{r}$ and $E_{s+r,u-r+1}^{r}$, respectively, are also in $\mathcal{S}$. Thus $E_{s,u}^{r}/Z_{s,u}^{r}$ and $B_{s,u}^{r}$ are in $\mathcal{S}$. It follows by the above exact sequences that if $E_{s,u}^{r+1}$ is in $\mathcal{S}$, then $E_{s,u}^{r}$ is in $\mathcal{S}$.

As we have $E_{s+r,u-r+1}^{r} = 0 = E_{s-r,u+r-1}^{r}$ for all $r \geq s+u+2$, we obtain $E_{s,u}^{\infty} = E_{s,u}^{s+u+2}$. To complete the proof, it is enough to show that $E_{s,u}^{\infty}$ is in $\mathcal{S}$.

There exists a finite filtration

$$0 = \phi^{-1}H_n \subseteq \phi^{0}H_n \subseteq \cdots \subseteq \phi^{n-1}H_n \subseteq \phi^{n}H_n = \operatorname{Tor}_{s-t}^{R}(N, X)$$

such that $E_{r,n-r}^{\infty} = \phi^{r}H_n/\phi^{r-1}H_n$ for all r, $0 \leq r \leq n$. Since $\operatorname{Tor}_{s-t}^{R}(N, X)$ is in $\mathcal{S}$, $\phi^{s}H_n$ is also in $\mathcal{S}$. Thus $E_{s,u}^{\infty} = \phi^{s}H_n/\phi^{s-1}H_n$ is in $\mathcal{S}$ as we desired. □

3. Applications

One can use Theorem 2.2 to study some sufficient conditions for finiteness of torsion functors of local cohomology modules. This is the subject of [10, Theorem 4.1] which shows that, for given integers s, t and given ideals $\mathfrak{a} \subseteq \mathfrak{b}$, $\operatorname{Tor}_{s}^{R}(R/\mathfrak{b}, \mathrm{H}_{\mathfrak{a}}^{t}(M))$ is finite whenever M is a finite R–module with $\dim_R(M) < \infty$, $\operatorname{Tor}_{s-t+i-1}^{R}(R/\mathfrak{b}, \mathrm{H}_{\mathfrak{a}}^{i}(M))$ is finite for all $i < t$, and $\operatorname{Tor}_{s-t+i+1}^{R}(R/\mathfrak{b}, \mathrm{H}_{\mathfrak{a}}^{i}(M))$ is finite for all $i > t$. In the following, we prove this theorem without assuming that M is finite and with no restrictions on dimension of M.

Corollary 3.1. (cf. [10, Theorem 4.1]) *Suppose that X and N are R–modules such that N is $\mathfrak{a}$–torsion. Assume also that s, t are non-negative integers such that*

(i) $\operatorname{Tor}_{s-t}^{R}(N, X)$ *is finite,*

(ii) $\operatorname{Tor}_{s-t+i-1}^{R}(N, \mathrm{H}_{\mathfrak{a}}^{i}(X))$ *is finite for all* i, $0 \leq i \leq t-1$, *and*

(iii) $\operatorname{Tor}_{s-t+i+1}^{R}(N, \mathrm{H}_{\mathfrak{a}}^{i}(X))$ *is finite for all* i, $t+1 \leq i \leq c$.

Then $\operatorname{Tor}_{s}^{R}(N, \mathrm{H}_{\mathfrak{a}}^{t}(X))$ *is finite.*

Proof. In Theorem 2.2, take $\mathcal{S}$ to be the subcategory of finite R–modules. The result follows. □

Let n be a positive integer and $\mathrm{H}_{\mathfrak{a}}^{i}(X)$ is in $\mathcal{S}$ for all $i > n$. In [2, Theorem 3.1], it is shown that $\mathrm{H}_{\mathfrak{a}}^{n}(X)/\mathfrak{a}\mathrm{H}_{\mathfrak{a}}^{n}(X)$ is in $\mathcal{S}$ whenever X is a weakly Laskerian R–module (i.e. the set of associated primes of any quotient module of X is finite) and X has finite Krull dimension. In the first part of the following result, we generalize the statement by removing all conditions on X.

Corollary 3.2. *Let* X *be an* R*–module and let* $\mathcal{S}$ *be a Serre subcategory of the category of* R*–modules such that, for a given integer* n, $\mathrm{H}_{\mathfrak{a}}^{i}(X)$ *is in* $\mathcal{S}$ *for all* $i > n$. *Assume that* N *is an* $\mathfrak{a}$*–torsion finite* R*–module and that* $\mathfrak{b}$ *is an ideal of* R *with* $\mathfrak{a} \subseteq \sqrt{\mathfrak{b}}$. *Then the following statements hold true.*

(i) *If* $n > 0$, *then* $N \otimes_R \mathrm{H}_{\mathfrak{a}}^{n}(X)$ *is in* $\mathcal{S}$. *In particular,* $\mathrm{H}_{\mathfrak{a}}^{n}(X)/\mathfrak{b}\mathrm{H}_{\mathfrak{a}}^{n}(X)$ *is in* $\mathcal{S}$.

(ii) *If* $n > 1$, *then* $\operatorname{Tor}_{1}^{R}(N, \mathrm{H}_{\mathfrak{a}}^{n}(X))$ *is in* $\mathcal{S}$. *In particular,* $\operatorname{Tor}_{1}^{R}(R/\mathfrak{b}, \mathrm{H}_{\mathfrak{a}}^{n}(X))$ *is in* $\mathcal{S}$.

Proof. Put $t = n$ in Theorem 2.2. For the first part take $s = 0$; and, for the second part, take $s = 1$. □

In the course of the remaining parts of the paper by $\operatorname{cd}_{\mathcal{S}}(\mathfrak{a}, X)$ ($\mathcal{S}$–cohomological dimension of X with respect to $\mathfrak{a}$) we mean the largest integer i in which $\mathrm{H}_{\mathfrak{a}}^{i}(X)$ is not in $\mathcal{S}$ (see [2, Definition 3.4] or [1, Definition 3.5]). If $\mathcal{S} = 0$, then $\operatorname{cd}_{\mathcal{S}}(\mathfrak{a}, X) = \operatorname{cd}(\mathfrak{a}, X)$ as in [7]. When $\mathcal{S}$ is the category of Artinian R–modules, we write $\mathrm{q}_{\mathfrak{a}}(X) := \operatorname{cd}_{\mathcal{S}}(\mathfrak{a}, X)$. Note that $\mathrm{q}_{\mathfrak{a}}(X) = \mathrm{q}(\mathfrak{a}, X)$ if R is local as in [5, Definition 3.1].

As an application of Corollary 3.2, we bring the following result which is essentially about non-finiteness of $\mathrm{H}_{\mathfrak{a}}^{\operatorname{cd}_{\mathcal{S}}(\mathfrak{a},X)}(X)$ where X is an arbitrary R–module. In [9, Theorem 3.2], it is shown that $\mathrm{H}_{\mathfrak{a}}^{\operatorname{cd}(\mathfrak{a},X)}(X)$ is not coatomic whenever $0 < \operatorname{cd}(\mathfrak{a}, X) = \operatorname{cd}(\mathfrak{a}, R/\operatorname{Ann}(X))$. In the second part of the following result, the equality condition is removed.

Corollary 3.3. *For an arbitrary* R*–module* X, *the following statements hold true.*

(i) *If* $\operatorname{cd}_{\mathcal{S}}(\mathfrak{a}, X) > 0$, *then* $\mathrm{H}_{\mathfrak{a}}^{\operatorname{cd}_{\mathcal{S}}(\mathfrak{a},X)}(X)/T$ *is not finite for any submodule* T *of* $\mathrm{H}_{\mathfrak{a}}^{\operatorname{cd}_{\mathcal{S}}(\mathfrak{a},X)}(X)$ *with* $T \in \mathcal{S}$. *In particular,* $\mathrm{H}_{\mathfrak{a}}^{\operatorname{cd}_{\mathcal{S}}(\mathfrak{a},X)}(X)$ *is not finite.*

(ii) *If* $\operatorname{cd}(\mathfrak{a}, X) > 0$, *then* $\mathrm{H}_{\mathfrak{a}}^{\operatorname{cd}(\mathfrak{a},X)}(X)/T$ *is not coatomic for any proper submodule* T *of* $\mathrm{H}_{\mathfrak{a}}^{\operatorname{cd}(\mathfrak{a},X)}(X)$. *In particular,* $\mathrm{H}_{\mathfrak{a}}^{\operatorname{cd}(\mathfrak{a},X)}(X)$ *is not coatomic.*

(iii) *If* $\mathrm{q}_{\mathfrak{a}}(X) > 0$, *then* $\mathrm{H}_{\mathfrak{a}}^{\mathrm{q}_{\mathfrak{a}}(X)}(X)/T$ *is not minimax for any Artinian submodule* T *of* $\mathrm{H}_{\mathfrak{a}}^{\mathrm{q}_{\mathfrak{a}}(X)}(X)$. *In particular,* $\mathrm{H}_{\mathfrak{a}}^{\mathrm{q}_{\mathfrak{a}}(X)}(X)$ *is not minimax.*

Proof. (i) Assume contrarily that $\mathrm{H}_{\mathfrak{a}}^{\operatorname{cd}_{\mathcal{S}}(\mathfrak{a},X)}(X)/T$ is finite. Then there exists an integer j such that $\mathfrak{a}^{j}(\mathrm{H}_{\mathfrak{a}}^{\operatorname{cd}_{\mathcal{S}}(\mathfrak{a},X)}(X)/T) = 0$; that is $\mathfrak{a}^{j}\mathrm{H}_{\mathfrak{a}}^{\operatorname{cd}_{\mathcal{S}}(\mathfrak{a},X)}(X) \subseteq T$. On the other hand, by Corollary 3.2, $\mathrm{H}_{\mathfrak{a}}^{\operatorname{cd}_{\mathcal{S}}(\mathfrak{a},X)}(X)/\mathfrak{a}^{j}\mathrm{H}_{\mathfrak{a}}^{\operatorname{cd}_{\mathcal{S}}(\mathfrak{a},X)}(X)$ is in $\mathcal{S}$ and so its quotient $\mathrm{H}_{\mathfrak{a}}^{\operatorname{cd}_{\mathcal{S}}(\mathfrak{a},X)}(X)/T$ is in $\mathcal{S}$. Therefore $\mathrm{H}_{\mathfrak{a}}^{\operatorname{cd}_{\mathcal{S}}(\mathfrak{a},X)}(X)$ is in $\mathcal{S}$ which contradicts the definition of $\operatorname{cd}_{\mathcal{S}}(\mathfrak{a}, X)$.

(ii) Assume that $\mathrm{H}_{\mathfrak{a}}^{\mathrm{cd}\,(\mathfrak{a},X)}(X)/T$ is coatomic. There exists a maximal submodule T'/T of $\mathrm{H}_{\mathfrak{a}}^{\mathrm{cd}\,(\mathfrak{a},X)}(X)/T$ so that there is an exact sequence

$$0 \longrightarrow T'/T \longrightarrow \mathrm{H}_{\mathfrak{a}}^{\mathrm{cd}\,(\mathfrak{a},X)}(X)/T \longrightarrow R/\mathfrak{m} \longrightarrow 0$$

for some maximal ideal $\mathfrak{m}$ of R, which results the exact sequence

$$T'/\mathfrak{a}T' + T \longrightarrow \mathrm{H}_{\mathfrak{a}}^{\mathrm{cd}\,(\mathfrak{a},X)}(X)/\mathfrak{a}\mathrm{H}_{\mathfrak{a}}^{\mathrm{cd}\,(\mathfrak{a},X)}(X) + T \longrightarrow R/\mathfrak{m} \longrightarrow 0$$

if one applies the functor $R/\mathfrak{a} \otimes_R -$. It can be seen either directly or deduced from Corollary 3.2 that $\mathrm{H}_{\mathfrak{a}}^{\mathrm{cd}\,(\mathfrak{a},X)}(X)/\mathfrak{a}\mathrm{H}_{\mathfrak{a}}^{\mathrm{cd}\,(\mathfrak{a},X)}(X) = 0$. Therefore its homomorphic image $\mathrm{H}_{\mathfrak{a}}^{\mathrm{cd}\,(\mathfrak{a},X)}(X)/\mathfrak{a}\mathrm{H}_{\mathfrak{a}}^{\mathrm{cd}\,(\mathfrak{a},X)}(X) + T$ is zero. This contradiction shows that $\mathrm{H}_{\mathfrak{a}}^{\mathrm{cd}\,(\mathfrak{a},X)}(X)/T$ is not coatomic.

(iii) Assume, in contrary, that $\mathrm{H}_{\mathfrak{a}}^{\mathrm{q}\,\mathfrak{a}(X)}(X)/T$ is a minimax module; so that there exists an exact sequence

$$0 \longrightarrow T'/T \longrightarrow \mathrm{H}_{\mathfrak{a}}^{\mathrm{q}\,\mathfrak{a}(X)}(X)/T \longrightarrow \mathrm{H}_{\mathfrak{a}}^{\mathrm{q}\,\mathfrak{a}(X)}(X)/T' \longrightarrow 0 \tag{3.1}$$

such that T'/T is finite and $\mathrm{H}_{\mathfrak{a}}^{\mathrm{q}\,\mathfrak{a}(X)}(X)/T'$ is Artinian. There is an integer j such that $\mathfrak{a}^j(T'/T) = 0$. As, by Corollary 3.2, $\mathrm{H}_{\mathfrak{a}}^{\mathrm{q}\,\mathfrak{a}(X)}(X)/\mathfrak{a}^j\mathrm{H}_{\mathfrak{a}}^{\mathrm{q}\,\mathfrak{a}(X)}(X)$ is Artinian its quotient $\mathrm{H}_{\mathfrak{a}}^{\mathrm{q}\,\mathfrak{a}(X)}(X)/\mathfrak{a}^j\mathrm{H}_{\mathfrak{a}}^{\mathrm{q}\,\mathfrak{a}(X)}(X) + T$ is also Artinian. Applying the functor $R/\mathfrak{a}^j \otimes_R -$ to the exact sequence (3.1) yields the exact sequence

$$\mathrm{Tor}\,_1^R(R/\mathfrak{a}^j, \mathrm{H}_{\mathfrak{a}}^{\mathrm{q}\,\mathfrak{a}(X)}(X)/T') \longrightarrow T'/T \longrightarrow \mathrm{H}_{\mathfrak{a}}^{\mathrm{q}\,\mathfrak{a}(X)}(X)/\mathfrak{a}^j\mathrm{H}_{\mathfrak{a}}^{\mathrm{q}\,\mathfrak{a}(X)}(X) + T$$

from which we obtain that T'/T is Artinian. Now, (3.1) implies that $\mathrm{H}_{\mathfrak{a}}^{\mathrm{q}\,\mathfrak{a}(X)}(X)/T$ is Artinian which contradicts with the fact that $\mathrm{H}_{\mathfrak{a}}^{\mathrm{q}\,\mathfrak{a}(X)}(X)$ is not Artinian. □

In [13, Proposition 3.1], it is proved that, for a positive integer n, $\mathrm{H}_{\mathfrak{a}}^i(X) = 0$ for all $i \geq n$ whenever X and all modules $\mathrm{H}_{\mathfrak{a}}^i(X)$, for all $i \geq n$, are finite and the ground ring R is local. In the following, among other things, we generalize this result for a general ring R and an arbitrary R–module X.

Corollary 3.4. *Let X be an arbitrary R–module and let n be a positive integer. Then the following statements are equivalent.*

(i) $\mathrm{H}_{\mathfrak{a}}^i(X)$ *is coatomic for all* $i \geq n$.
(ii) $\mathrm{H}_{\mathfrak{a}}^i(X)$ *is finite for all* $i \geq n$.
(iii) $\mathrm{H}_{\mathfrak{a}}^i(X) = 0$ *for all* $i \geq n$.

Proof. (i) ⇔ (iii). This is clear from Corollary 3.3(ii).

(ii) ⇔ (iii). It follows from Corollary 3.3(i). □

In consistence of Corollary 3.4, one can state the following result about Artinian-ness of local cohomology modules from a point upward.

Corollary 3.5. *Let X be an arbitrary R–module and let n be a positive integer. Then the following statements are equivalent.*

(i) $\mathrm{H}_{\mathfrak{a}}^i(X)$ *is minimax for all* $i \geq n$.
(ii) $\mathrm{H}_{\mathfrak{a}}^i(X)$ *is Artinian for all* $i \geq n$.

Proof. This follows from Corollary 3.3(iii). □

Acknowledgement. The authors would like to thank the referee for her/his comments.

References

1. M. Aghapournahr, L. Melkersson, *Local cohomology and Serre subcategories*, J. Algebra, **320** (2008), 1275–1287.
2. M. Asgharzadeh, M. Tousi, *A unified approach to local cohomology modules using serre classes*, arXiv: 0712.3875v2 [math.AC].
3. M. P. Brodmann, R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics, 60. Cambridge University Press, Cambridge, 1998.
4. D. Delfino, T. Marley, *Cofinite modules of local cohomology*, J. Pure Appl. Algebra, **121**(1) (1997), 45–52.
5. M. T. Dibaei, S. Yassemi, *Associated primes and cofiniteness of local cohomology modules*, manuscripta math., **117** (2005), 199–205.
6. A. Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)*, North-Holland, Amsterdam, 1968.
7. R. Hartshorne, *Cohomological dimension of algeraic varieties*, Ann. of Math., **88** (1968), 403–450.
8. R. Hartshorne, *Affine duality and cofiniteness*, Invent. Math., **9** (1970), 145–164.
9. M. Hellus, *On the associated primes of Matlis duals of local cohomology modules II*, arXiv: 0906.0642v2 [math.AC].
10. K. Khashyarmanesh, *On the finiteness properties of extention and torsion functors of local cohomology modules*, Proc. Amer. Math. Soc., **135** (2007), 1319–1327.
11. L. Melkersson, *Modules cofinite with respect to an ideal*, J. Algebra, **285** (2005), 649–668.
12. J. Rotman, *An introduction to homological algebra*, Academic Press, (1979).
13. K.-I. Yoshida, *Cofiniteness of local cohomology modules for ideals of dimension one*, Nagoya Math. J., **147** (1997), 179–191.
14. H. Zöschinger, *Minimax Moduln*, J. Algebra, **102** (1986), 1–32.
15. H. Zöschinger, *Koatomare Moduln*, Math. Z., **170** (1980), 221–232.

[1] Faculty of Mathematical Sciences and Computer, Tarbiat Moallem University, Tehran, Iran; and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran.

E-mail address: `dibaeimt@ipm.ir`

[2] Payame Noor University (PNU), Iran.

E-mail address: `vahidi.ar@gmail.com`